approximation, these conditions will be reduced to the solvability of the linear equation

$$Pc_{j-2,2} + P_j = 0 \quad (j = 4, 5, \ldots)$$

where $C_{j-2,2}$ is the integration constant of the solution z_{j-2} of the $(j-2)$-th approximation respectively. The coefficient $P \neq 0$ will remain general for all j-th approximations.

Thus we have the following theorem.

Theorem 5. Eq.(1) has two families of 4π-periodic solutions which are represented by the first series of (2), and the first approximation has the form

$$z_1 = \pm\gamma(-\alpha e^{-iE/2} + e^{iE/2})$$

Similar result can be obtained for the second libration point L_5.

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GROUP-THEORETICAL ANALYSIS OF THE EQUATIONS OF MOTION OF A THIXOTROPIC FLUID*

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A group-theoretical classification of a system of equations describing a one-dimensional flow of a thixotropic fluid is carried out. Certain invariant solutions are analysed.

1. We shall regard, as a thixotropic fluid, a medium in which an increase in shear stresses leads to a decrease in viscosity due to disruption of internal structure of the medium. Such fluids include asphalt - and paraffin-contg. oils, a number of polymer solutions, clayey solutions, et al.

We describe the flow of a thixotropic fluid, of viscosity μ depending on a single structural dimensionless parameter λ, with help of the models /1, 2/ which can be written, in the one-dimensional case, in the form

$$u_t = (\mu(\lambda) u_x)_x, \quad \lambda_t = \Phi(\lambda, u_x) \tag{1.1}$$

Models of this type are also used when describing the filtration of a viscoelastic fluid /3/.

Let us investigate the group properties /4, 5/ of system (1.1).

When the functions $\mu(\lambda)$ and $\Phi(\lambda, u_x)$ are arbitrary, system (1.1) admits of a three-dimensional algebra L_3 of infinitesimal operators with the basis $X_1 = \partial/\partial t$, $X_2 = \partial/\partial x$, $X_3 = \partial/\partial u$, corresponding to the shears in t, x, u.

Let us determine under what special conditions imposed on μ and Φ this algebra can be extended. The analysis of the system of defining equations /4/ for (1.1) shows that the following assertion holds:

If $\mu' \not\equiv 0$, $\partial\Phi/\partial u_x \not\equiv 0$, $\partial\Phi/\partial\lambda \not\equiv 0$, then the algebra extends only for one of the following sets of μ and Φ: 1) μ is an arbitrary function, $\Phi = u_x^{\alpha} f(\lambda)$, where f is an arbitrary function, $\alpha \neq 0$; and the positive basis operator has the form

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$$X_4 = \alpha x \frac{\partial}{\partial x} + 2\alpha t \frac{\partial}{\partial t} + (\alpha - 2) u \frac{\partial}{\partial u}$$

2) μ is an arbitrary function, $\Phi = (\mu^{1+\gamma}/\mu') F(u_x \mu^\beta)$, where $\partial F/\partial u_x \not\equiv 0, \beta, \gamma \in R$; and the additional operator is

$$X_5 = \frac{1-\gamma}{2} x \frac{\partial}{\partial x} - \gamma t \frac{\partial}{\partial t} + \left(\frac{1-\gamma}{2} - \beta\right) u \frac{\partial}{\partial u} + \frac{\mu}{\mu'} \frac{\partial}{\partial \lambda}$$

3) μ is an arbitrary function, $\Phi = (\mu/\mu')(1 + \mu u_x^\varepsilon)$ where $\varepsilon \neq 0$, and the additional operators are

$$X_6 = \varepsilon x \frac{\partial}{\partial x} + (\varepsilon - 2) u \frac{\partial}{\partial u} + 2\varepsilon \frac{\mu}{\mu'} \frac{\partial}{\partial \lambda}, \quad X_7 = e^{-t} \frac{\partial}{\partial t} + e^{-t} \frac{\mu}{\mu'} \frac{\partial}{\partial \lambda}$$

2. Let us investigate, for the cases shown above in Sect.1, the invariant solutions connected with the appearance of additional basis operators. To do this we write, for each of these cases, the invariant solutions corresponding to the operators which appear in the optimal system of subalgebras /4/ and contain the additional operators.

1) The optimal system of subalgebras contains the operator $X_4 + cX_3$, $c \in R$, and if $\alpha \neq 2$, then $c = 0$. The corresponding invariant solution has the form

$$u = c \ln t + t^{-\nu} \varphi(\xi), \ \lambda = \psi(\xi), \ \xi = xt^{-1/2}$$

$\nu = (2 - \alpha)/(2\alpha)$, and φ and ψ satisfy the following system of ordinary differential equations:

$$\nu\varphi + {}^1/_2\xi\varphi' + (\mu(\psi)\varphi')' = {}^1/_4 c, \ {}^1/_2\xi\psi' + f(\psi)\psi'^{\alpha} = 0$$

2) If $\gamma \neq 0$ or $\gamma \neq 1$, the operator appearing in the optimal system has the form $X_5 + cX_3$, $c \in R$, and $c = 0$ when $\sigma = \beta - {}^1/_2(1 - \gamma) \neq 0$. The invariant solution $u = (c/\gamma)\ln t + \varphi(\xi)$, $\mu(\lambda) = t^{-1/\gamma}\psi(\xi)$, $\xi = xt^{(1-\gamma)/(2\gamma)}$, and φ and ψ satisfy the system

$$\frac{c}{\gamma} + \frac{\sigma}{\gamma}\varphi + \frac{1-\gamma}{2\gamma}\xi\varphi' = (\psi\varphi')', \quad (1-\gamma)\xi\psi' = 1 + \gamma\psi^{1+\gamma} F(\varphi'\psi^\beta)$$

When $\gamma = 0$, the optimal system contains $X_5 + aX_1 + cX_3$, $a, c \in R$, and if $\beta \neq {}^1/_2$, then $c = 0$. The invariant solution is

$$u = x^{1-2\beta}\varphi(\xi) + 2c \ln x, \quad \mu(\lambda) = x^2\varphi(\xi), \quad \xi = t - 2a \ln x$$

When $\gamma = 1$, the required operator has the form $X_5 + bX_2 + cX_3$ (when $\beta \neq 0$, then $c = 0$), and the corresponding solution is

$$u = t^\beta\varphi(\xi) - c \ln t, \ \mu(\lambda) = t^{-1}\psi(\xi), \ \xi = x + b \ln t$$

The functions φ and ψ satisfy, in each case, the corresponding system of differential equations.

3) The operators appearing in the optimal system can be conveniently written in the form: a) $aX_1 + cX_3 + X_6$, b) $bX_2 + dX_3 + X_7$ and c) $cX_3 + X_6 + X_7$, where $a, b, c, d \in R$, and $c = 0$ when $\varepsilon \neq 2$. The invariant solutions for each of these operators are written in the form

$$\begin{aligned}
&\text{a)}\ u = x^{(\varepsilon-2)/\varepsilon}\varphi(\xi) + 2c \ln x, \quad \mu(\lambda) = x^2\psi(\xi), \quad \xi = x^{-2a}e^t \\
&\text{b)}\ u = de^t + \varphi(\xi), \quad \mu(\lambda) = e^t\psi(\xi), \quad \xi = x - be^t \\
&\text{c)}\ u = e^{(\varepsilon-2)\tau}\varphi(\xi), \quad \mu(\lambda) = e^{2\varepsilon\tau}\tau\psi(\xi), \quad \xi = \ln x - \varepsilon\tau, \quad \tau = e^t
\end{aligned}$$

Here, as before, the functions φ and ψ satisfy the corresponding systems of differential equations.

3. Let us analyse some of the invariant solutions obtained in Sect.2.

A kinetic equation linear in λ and of some degree in u_x, i.e. an equation of the form $\lambda_t + \lambda = u_x^m$, is often used in practice. A power relation $\mu = \lambda^n$ is assumed in Eqs.(1.1), and in this case the invariant solution can be written thus:

$$u = (x_0 - x)^p\varphi(t), \ \lambda = (x_0 - x)^{2/n}\psi(t), \ p = (2/(mn)) + 1, \ x_0 > 0$$

In the case when $n = -1$, the system of differential equations in φ and ψ can be integrated in quadratures, and its solution can be written in the form

$$t = t_0 + \ln\left(1 + l\int_{\varphi_0}^{\varphi} \exp[q(\varphi^m - \varphi_0^m)]\, d\varphi\right) \tag{3.1}$$

$$\psi = \psi_0 \exp [q (\varphi^m - \varphi_0{}^m)] \left(1 + l \int\limits_{\varphi_0}^{\varphi} \frac{\exp (q \varphi^m)}{\varphi} \, d\varphi \right)^{-1}$$

$$q = \frac{(m-2)^{m-1}}{2(m-1) m^{m-1}}, \quad l = \frac{\psi_0 m^2}{(m-1)(m-2)}$$

In the case of a thixotropic medium, the quantity m must be positive. Then from (3.1) we see that if a zone with a disrupted structure $(0 \leqslant x \leqslant x_0)$ did exist at the initial instant, then at subsequent instances a perturbation in a specified mode at the boundary $x = 0$ would not propagate beyond the boundary $x = x_0$.

In the case of the operator $bX_2 + dX_3 + X_7$, the solvability in quadratures of the system of equations for φ and ψ will make it possible to write the corresponding invariant solution in explicit form

$$u = de^t + b\psi^{-1}(\xi) + c_3, \quad \mu(\lambda) = e^t \psi(\xi)$$

$$\psi = \exp [-{}^1/_2 \, db^{-1} (\xi + c_1)^2] \left(b \int \exp [{}^1/_2 \, db^{-1} (\xi + c_1)^2] \, d\xi + c_2 \right)$$

$$\xi = x - be^t$$

where c_1, c_2, c_3 are arbitrary constants.

In the case when a system of differential equations in φ and ψ cannot be solved in quadatures, then the corresponding invariant solutions can be analysed using numerical methods and the methods of analytic theory of ordinary differential equations.

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